

The effect of spillovers and congestion on the endogenous formation of jurisdictions*

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April 3, 2014

Abstract

This paper analyzes the effect of inter-jurisdictional spillovers and congestion of local public services on the segregative properties of endogenous formation of jurisdictions. Households choosing to live at the same place form a jurisdiction, the aim of which is to produce congested local public services, resulting in generate positive spillovers to other jurisdictions. In every jurisdiction, the production of the local public services is financed through a local tax based on households' wealth. Local wealth tax rates are democratically determined in every jurisdiction. Households consume the available amount of public services in their jurisdiction and a composite private good. Any household is free to leave its jurisdiction for another that would increase its utility. A necessary and sufficient condition to have every stable jurisdiction structure segregated by wealth is identified: the public services must be either a gross substitute or a gross complement to the private good.

JEL Classification: C78; D02; H73; R13

Keywords: Jurisdictions; Segregation; Spillovers; Congestion

*I am indebted to Richard Arnott, Francis Bloch, Pierre-Philippe Combes, Nicolas Gravel and two anonymous referees for their very helpful comments and suggestions. I really appreciated Myrna Wooders's comments on a previous article that underlined the importance of using a discrete set of households instead of a continuum in local public goods economies, as it gave me the idea to work on this new paper. I am also very grateful to the participants of the Association for the Public Economic Theory 2012 - in particular Gareth Myles and John Conley-, and participants of the Regional Science Council 2010.

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1 Introduction

There exists a widely-held belief that metropolitan sectors are wealth-segregated. Rich and poor municipalities co-exist within the same urban area. The article investigates how congestion effects and positive spillovers affect segregation.

According to Tiebout's intuitions[11], individuals choose their place of residence according to a trade-off between local tax rates and the amounts of public services provided, which leads every jurisdiction to be homogeneous. The formation of a jurisdiction structure is endogenous, due to the free mobility of households, that can "vote with their feet"; that is to say, they can leave their jurisdiction for another one, if they are unsatisfied with that jurisdiction's tax rate and amount of public services. Westhoff [12] was among the first economists to provide a formal model based on Tiebout's intuitions. In this model, households can enjoy two different goods: a local public good, financed through a local tax on wealth, which is a pure club good (only households living in the jurisdiction that produced it can consume the local public good that does not suffer from the congestion effect), and a composite private good, which amount is equal to the after-tax wealth. Westhoff found a condition that ensures the existence of an equilibrium: the slopes of individuals' indifference curves must be ordered with respect to the private wealth in the tax rate-amount of public good space. If this condition is respected, not only will an equilibrium exist, but, at equilibrium, the jurisdictions structure will be segregated, as per the definition provided by Ellickson [5].

Gravel and Thoron [7] identified a necessary and sufficient condition that ensures the segregation of every stable jurisdiction structure: the public good must, for all levels of prices and wealth, either always be a complement or always be a substitute to the private good. This condition is called the Gross Substitutability/Complementarity (GSC) condition. This condition is equivalent to the preferred tax rate being a monotonous function of private wealth, for any level of prices and wealth. Biswas, Gravel and Oddou [1] integrated a welfarist central government to the model, the purpose of which is to maximize a generalized utilitarian social welfare function by implementing an equalization payment policy. They found that the GSC condition remains necessary and sufficient. That is also the case when a land market is introduced (see Gravel and Oddou[6]).

However, none of these models allows for inter-jurisdictional spillovers nor congestion effects. To be convinced of the existence of inter-jurisdictional spillovers, one can simply consider the example of an international airport in a suburban town (Newark, for instance). Every municipality located nearby this town would benefit from the airport, though it would not be located on its administrative territory. The most striking proof of how congestion effects existence is the traffic jam : the benefit households will gain from a public road is clearly affected by the number of its users.

As provided by Epple & al. [9], empirical evidences suggest that households' mobility leads to segregation when peer-effects are taken into account. One can notice that the calibration of households' preferences respects the GSC condition.

Wooders ([13] and [14]) and Conley[4] provided several generalizations of Westhoff's model, by relaxing some assumptions such as the non-rivalry of the local public goods. They assumed that the public good suffers from congestion (called here "crowding effect"), which means that households' utility decreases with the number of people that consume the public good, for any given amount of produced public good. Those papers are closer to Tiebout's intuitions in the sense that crowding effect are allowed in Tiebout's paper to justify his assumption that jurisdictions would not be too crowded.

Allowing for crowding effects is likely to affect the segregative characteristic of the equilibrium, as presumably they would reinforce the segregative properties of endogenous jurisdiction formation. As Wooders writes [15]: *"in general, even in the crowding types model, where it has the best chance, players do not necessarily sort into taste homogeneous jurisdictions."*

Nechyba [8] developed a model within a discrete set of households. In his model, spillovers between jurisdictions were allowed, because a household's utility depends not only on the amounts of local public good provided by its jurisdiction and of the national public good, but also on the amounts of public good provided by all other jurisdictions. After having ensured the existence of an equilibrium under certain conditions, Nechyba identified sufficient conditions for a stable jurisdiction structure to be segregated. Unfortunately, one of these sufficient conditions was the absence of spillovers between jurisdictions, which is a pretty strong assumption that might not be necessary.

A major contribution provided by those articles is the assumption that the set of households is discrete rather than continuous. One may reasonably assert that real economies have finite numbers of consumers, and, so far, no satisfying evidences that all existing economies with a local public good can be rigorously represented with a continuum of households. Thus, a discrete set of households strengthen the credibility of the model. Furthermore, assuming a finite number of households rather than a continuum of agents is more intuitive when the existence of congestion effects is allowed. A main difference with Gravel & Thoron [7] is that the current paper has a finite number of consumers.

The effect of spillovers on the provision of public goods and on the equilibrium has been analyzed by Bloch and Zenginobuz [3] and [2]. However, the consequences in terms of segregation that the existence of spillovers may imply are not examined.

The article is organized as follows. Section 2 introduces the formal model. Section 3 provides an example of how congestion and spillovers can modify a jurisdiction structure. Section 4 states and proves the results. Finally, section 5 concludes.

2 The formal model

We consider a model *à la* Gravel & Thoron, improved by the existence of spillovers and congestion effects. An economy is composed of six elements. The first element, which is an important difference from Gravel & Thoron's article [7], is a discrete set of households $N \subset \mathbb{N}$, where $\text{card}(N) = n \in \mathbb{N}$.

The second element is a wealth distribution, modelled as a function $\omega : N \rightarrow \mathbb{R}_+$ - household i is endowed with a wealth $\omega_i \in \mathbb{R}_+$ - with ω being an increasing and bounded from the above function.

The third element is a specification of households' preferences. Households have identical preferences, represented by a twice differentiable, increasing and concave utility function

$$U : \begin{cases} \mathbb{R}_{++}^2 & \longrightarrow \mathbb{R}_+ \\ (Z, x) & \longmapsto U(Z, x) \end{cases}$$

where

1. Z is the available amount of public services households can enjoy,
2. x is the amount of a composite private good.

We denote \mathbb{U} as the set of all functions satisfying the properties defined above.

The fourth element is a finite and discrete set of possible locations. Each household has to choose a place of residence among all the conceivable locations. $\mathbb{L} \subset \mathbb{N}$ is the

finite set of locations. The possibility for some locations to be empty is not ruled out. Households living at the same location form a jurisdiction. We denote $J \subseteq \mathbb{L}$ the set of jurisdictions, with $\text{card}(J) = M$. The set of households living in jurisdiction j is denoted $N_j \subseteq N$, with $\text{card}(N_j) = n_j \leq n$

The fifth element is a matrix of spillovers coefficients. Since local public services create spillovers in other jurisdictions, the total amount of public services a household in jurisdiction j can enjoy (Z_j) depends not only on the available amount of public services produced by jurisdiction j , but also on the amount of public services produced by the other jurisdictions. This amount is given by

$$Z_j = \zeta_j + S_j$$

with :

- ζ_j is the available amount of public services produced by jurisdiction j ,
- S_j is the amount of spillovers from other jurisdictions in jurisdiction j , given by

$$S_j = \sum_{k \in J - \{j\}} \beta_{jk} \zeta_k$$

where $\beta_{jk} \in \mathbb{R}_+$ represents the spillovers coefficient of jurisdiction k 's local public services in jurisdiction j .

Let us denote B as the square matrix of order M , that represents the spillovers coefficients, with $\forall (j, j') \in J^2, \beta_{jj'} \in [0; 1]$. No assumption needs to be made on B . It is exogenously determined¹. We assume that, $\forall j \in J, \beta_{jj} = 1$, so that one can rewrite the available amount of public services in jurisdiction j as: $Z_j = \sum_{k \in J} \beta_{jk} \zeta_k$

The matrix B is not necessarily symmetric. We denote \mathbb{B} as the set of all square matrix of order M such that, $\forall (j, j') \in J^2, \beta_{jj'} \in [0; 1]$ and $\beta_{jj} = 1$.

Finally, the sixth element is a congestion parameter. The amount of available local public services *produced* by jurisdiction j is given by

$$\zeta_j = \frac{t_j \varpi_j}{\left(\sum_{k \in J} \beta_{kj} n_k \right)^\alpha}$$

with :

- t_j being the local tax rate,
- $\varpi_j = \sum_{i \in N_j} \omega_i$ being the aggregate wealth in j ,
- $\alpha \in \mathbb{R}_+$, being the congestion parameter in jurisdiction j .

Assuming that the intensity of the congestion faced by jurisdiction j 's public services caused by the number of households in jurisdiction j' depends on the spillovers coefficient j 's public services created in j' is quite reasonable, since public services would be "more consumed" by households in a jurisdiction having a high spillovers coefficient than by households that receive little spillovers from this public good. Moreover, it is assumed that if jurisdiction j 's public services create no spillovers in jurisdiction j' (i.e. $\beta_{j'j} = 0$), then the congestion function is constant with respect to $n_{j'}$.

¹For instance, spillovers coefficients can represent the distance between two jurisdictions (the closer jurisdiction j is to jurisdiction j' , the closer to 1 will be $\beta_{jj'}$), or be affected by political agreements concluded between jurisdictions...

Definition 1. A economy is composed of six elements:

- A set of households $N \subset \mathbb{N}$
- A wealth distribution ω
- Preferences represented by the utility function $U \in \mathbb{U}$
- A set of locations $\mathbb{L} \subset \mathbb{N}$
- A spillovers coefficient matrix $B \in \mathbb{B}$
- A congestion parameter $\alpha \in \mathbb{R}_+$

We denote Δ as the set of all conceivable economies.

For simplicity, we denote $F_j = \frac{\varpi_j}{(\sum_{k \in J} \beta_{kj} n_k)^\alpha}$ as jurisdiction j 's fiscal potential; that is to say the maximal available amount of public services jurisdiction j can produce (if $t_j = 1$).

The local tax rate is determined according to a democratic rule, in the sense that a jurisdiction's tax rate must be the favorite tax rate of one of its household. Hence, every household has to determine its favorite tax rate, denoted $t^* : \mathbb{R}_+^3 \rightarrow [0; 1]$, which depends on:

- the fiscal potential F ,
- the spillovers from other jurisdictions' public services S (taken as given),
- the private wealth ω_i .

Formally,

$$t^*(F, S, \omega_i) \in \arg \max_{t \in [0; 1]} U(tF + S, (1 - t)\omega_i) \quad (1)$$

Lemma 1. For all utility functions belonging to \mathbb{U} , $\forall (F, S, \omega_i) \in \mathbb{R}_+^3$, preferences are single-peaked with respect to t , so $t^*(F, S, \omega_i)$ exists (see proof in Appendix 6.1).

Definition 2. A jurisdiction structure is a vector $\Omega = (J, \{I_j\}_{j \in J}; (\{t_j\}_{j \in J}); (\{S_j\}_{j \in J}))$.

We define a jurisdiction structure as a set of jurisdictions, the repartition of households among the jurisdictions, the vector of tax rates implemented by every jurisdiction and the vector of spillover amounts received by the different jurisdictions.

Definition 3. A jurisdiction structure $\Omega = (J, (\{I_j\}_{j \in J}); (\{t_j\}_{j \in J}); (\{S_j\}_{j \in J}))$ is **stable** in the economy $(N, \omega, U, \mathbb{L}, B, \alpha)$ if and only if

1. $\forall j, j' \in J, \forall i \in I_j, U(Z_j, (1 - t_j)\omega_i) \geq U(Z_{j'}, (1 - t_{j'})\omega_i)$,
2. $\forall j \in J, (\sum_{k \in J} \beta_{kj} n_k)^\alpha \zeta_j = t_j \varpi_j$,
3. $\forall j \in J, \exists i \in I_j : t_j = t^*(F_j, S_j, \omega_i)$.

Literally, a jurisdiction structure is stable if:

1. No household can increase its utility by modifying its consumption bundle or by leaving its jurisdiction,
2. Every jurisdiction's budget is balanced,
3. The tax rate is democratically determined in every jurisdiction.

Regarding the first point of the definition, it is important to mention that households are assumed to be "myopic", e.g. they do not consider the impact on tax rates and amounts of public good vectors that their own relocation would generate. This assumption is used in many articles, such as [13].

We can now express formally the definition of the segregation, which is the same definition as in [12].

Definition 4. A jurisdiction structure $\Omega = (J, (\{I_j\}_{j \in J}); (\{t_j\}_{j \in J}); (\{S_j\}_{j \in J}))$ in the economy $(N, \omega, U, \mathbb{L}, B, \alpha)$ is **segregated** if and only if $\forall h, i, k \in N$ such that $\omega_h < \omega_i < \omega_k$, $(h, k) \in I_j$ and $i \in I_{j'}$ $\Rightarrow Z_j = Z_{j'}$ and $t_j = t_{j'}$

In other words, a jurisdictions structure is wealth-segregated if, except for groups of jurisdictions offering the same available amount of public services and implementing the same tax rate, the poorest household of a jurisdiction with a high per capita wealth is (weakly) richer than the richest household in a jurisdiction with a lower per capita wealth.

The next section introduces an example showing how the existence of spillovers and congestion affects a jurisdiction structure at equilibrium.

3 Example

This section presents an example of an economy, where congestion and spillovers are introduced in turn, so as to examine their impact on the jurisdiction structure at the equilibrium. This example shows how the introduction of congestion and spillovers affects the jurisdiction structure; congestion seems to favor segregation, while spillovers mitigate it.

In this example, we start from a situation where the jurisdiction structure is stable and non-segregated. We first assume that public services do not generate spillovers, nor suffer from congestion. Then, we introduce congestion effects, which will lead to instability. The new jurisdiction structure will then be segregated. Finally, we consider that local public services generate spillovers; that the jurisdiction structure will no longer be stable, and the first jurisdiction structure will arise. This example suggests that congestion effects increase the segregative properties of endogenous jurisdictions formation, while the presence of spillovers mitigates them.

Let us consider the example provided by Gravel & Thoron. Households' preferences are represented by

$$U(Z, x) = \begin{cases} \ln(Z) + 4x - x^2 & \text{if } x \leq 2 \\ \ln(Z) & \text{otherwise} \end{cases}$$

Such an utility function is continuous, twice differentiable, weakly increasing and concave with respect to every argument. Consider an economy with two jurisdictions j_1 and j_2 and three types of households a, b, c with private wealth $\omega_a = 2 - \sqrt{2}$, $\omega_b = 1.5$ and $\omega_c = 2$, with $n_a = 897,000$, $n_b = 300,000$ and $n_c = 100$.

As long as $x \leq 2$, the preferred tax rate function is given by

$$t^*(F, S, \omega_i) = \frac{\omega_i - 2 + \sqrt{(\omega_i - 2)^2 + 2}}{2\omega_i}$$

Here, the favorite tax rate function depends only on the private wealth, and not on the fiscal potential or on the amount of spillovers, which will greatly simplify the example. Determining the preferred tax rate function if $x > 2$ will not be required, no household will be endowed with more than 2 units of private good.

Let us assume first that there is no congestion and no spillovers. Then, the available amount of public services in a jurisdiction j is simply the tax revenue: $Z_j = t_j \varpi_j$. A stable jurisdiction structure would be households of type a and c living in j_1 , and households of type b living in j_2 (see Appendix 6.2).

Now, let us reconsider the example when the local public services suffer from congestion, with $\alpha = 0.2$. So, for $j = \{j_1, j_2\}$, one has $Z_j = \frac{t_j \varpi_j}{n_j^{0.2}}$.

The current jurisdiction structure will then become unstable, and the new stable jurisdiction structure will be segregated: jurisdiction 1 will be composed of households of type a , while households of types 2 and 3 will be better-off in jurisdiction 2 (see Appendix 6.3).

In this very specific example, the congestion seems to increase the segregative properties of the endogenous jurisdiction structure formation.

Let us now introduce spillovers in the example in order to observe what impact they can have. Suppose that jurisdiction j_1 's local public services generate spillovers in jurisdiction j_2 , and vice-versa. The available amount of public services in a jurisdiction j is then given by

$$Z_j = \frac{t_j \varpi_j}{(n_j + \beta_{j'j} n_{j'})^{0.2}} + \beta_{jj'} \frac{t_{j'} \varpi_{j'}}{(n_{j'} + \beta_{jj'} n_j)^{0.2}}$$

Although j_1 produces more public services than j_2 , suppose that jurisdiction j_1 receives more spillovers from j_2 's public services than vice-versa²: $\beta_{12} = 0.9$ and $\beta_{21} = 0.3$.

Again, the segregated jurisdiction structure will become unstable, and the previous non-segregated jurisdiction structure will reappear and be stable (see Appendix 6.4).

As a conclusion, this example suggests that congestion favors the segregative properties of endogenous jurisdiction formation, whereas the existence of spillovers tends to decrease the number of stable segregated jurisdiction structures.

However, in the next section, the validity of the GCS condition, that was necessary and sufficient to ensure the segregation of every stable jurisdictions structure, is established within the existence of congestion effect and spillovers.

4 Results

The main result of this paper is the robustness of Gravel and Thoron's results to the existence of spillovers and congestion effects, and to the introduction of a discrete set of households rather than a continuum. The Gross Substitutability/Complementarity (GSC) condition, as defined below, is necessary and sufficient to ensure the existence of any stable jurisdiction structure. As in Gravel and Thoron's article, this condition is equivalent to the monotonicity of the preferred tax rate function with respect to the private wealth, for any given amount of the other arguments. To prove this equivalence, let us first establish the following lemma.

Lemma 2. $\forall U \in \mathbb{U}, \forall (F, S, \omega_i) \in \mathbb{R}_+^3$, the preferred tax rate function is a monotonic function of the Marshallian demand for the public good:

$$t^*(F, S, \omega_i) \equiv \frac{Z^M(\frac{1}{F}, \frac{1}{\omega_i}, 1 + \frac{S}{F}) - S}{F} \quad (2)$$

²For instance, j_1 may have implemented a restrictive policy in order to prevent households living in j_2 from congesting its public services.

Proof. At the optimum, the Marginal Rate of Substitution (MRS) is equal to the price ratio. Then:

$$\frac{U_Z(Z^M(p_Z, p_x, R), x^M(p_Z, p_x, R))}{U_x(Z^M(p_Z, p_x, R), x^M(p_Z, p_x, R))} = \frac{p_Z}{p_x} \quad (3)$$

where $Z^M(p_Z, p_x, R)$ and $x^M(p_Z, p_x, R)$ are respectively the Marshallian demands for public services and for the private good when the public services price is p_Z , the private good price is p_x and the income is R^3 . The FOC of the utility maximization program with respect to t (1) implies that:

$$\frac{U_Z(t^*F + S, (1 - t^*)\omega_i)}{U_x(t^*F + S, (1 - t^*)\omega_i)} = \frac{\omega_i}{F} \quad (4)$$

Then, using (3) and (4), we know that:

$$t^*(F, S, \omega_i)F + S \equiv Z^M(p_Z, p_x, R) \quad (5)$$

when $p_Z = \frac{1}{F}$, $p_x = \frac{1}{\omega_i}$ and $R = 1 + \frac{S}{F}$.

$$t^*(F, S, \omega_i)F - S \equiv Z^M\left(\frac{1}{F}, \frac{1}{\omega_i}, 1 + \frac{S}{F}\right) \quad (6)$$

which leads to (2). \square

This lemma states that the favorite tax rate function is equivalent to an increasing linear function of the Marshallian demand for public services. This lemma is used to prove that the favorite tax rate function is monotonic with respect to the private wealth if, and only if, the public services are either always a substitute or a complement to the private good. This condition is called the Gross Substitutability Complementarity (GSC) condition.

Definition 5. *If the GCS condition holds, then, $\forall (p_Z, p_x, R) \in \mathbb{R}_+^3$, one has either $\frac{\partial Z^M(p_Z, p_x, R)}{\partial p_x} \leq 0$ (if public services are a gross complement to the private good) or $\frac{\partial Z^M(p_Z, p_x, R)}{\partial p_x} \geq 0$ (if public services are a gross substitute to the private good)*

Lemma 3. *For all utility functions belonging to \mathbb{U} , the favorite tax rate function is always monotonic with respect to private wealth if, and only if, the public services are a gross substitute or a gross complement to the private good.*

Proof. To prove this lemma, we will show that the derivative of the preferred tax rate function with respect to the private wealth can be expressed as a negative function of the derivative of the Marshallian demand for the public services with respect to the price of the private good. Consequently, the preferred tax rate function will be monotonic with respect to the private wealth if, and only if, the Marshallian demand for public services are monotonic with respect to the composite private good price.

By deriving (2) with respect to the private wealth, one gets :

$$\frac{\partial t^*(F, S, \omega_i)}{\partial \omega_i} = \frac{-1}{F\omega_i^2} \frac{\partial Z^M(\bar{p}_Z, p_x, R)}{\partial p_x}$$

so

$$\text{sign}\left(\frac{\partial t^*(F, S, \omega_i)}{\partial \omega_i}\right) = -\text{sign}\left(\frac{\partial Z^M(\bar{p}_Z, p_x, R)}{\partial p_x}\right)$$

\square

³The Marshallian demands must be interpreted as a dual representation of the preferences, and not as the amount of public services a household would "buy"

The GSC condition is restrictive enough to be discussed, but nevertheless it is not outlandish. For instance, suppose the only competence the central government has transferred to the studied jurisdictions level is social aid. Then, one may assume that the public services will be a substitute for the private good. On the contrary, if the jurisdiction is competent only in cultural activities, then the public services will probably be a complement to the private good. Now, suppose that the jurisdiction is in charge of primary schools. In this case, the relation between the public services and the other goods is not trivial, and may vary with respect to the jurisdiction's fiscal potential and amount of received spillovers, and with respect to the private wealth.

To prove the sufficiency of the GSC condition, the notion of the indifference curve has to be introduced.

Definition 6. $\forall(t, \bar{u}, S, \omega_i) \in [0; 1] \times \mathbb{R}_+^3$, let us define $F^{\bar{u}}(t, S, \omega_i)$:

$$U(tF^{\bar{u}}(t, S, \omega_i) + S), (1-t)\omega_i \equiv \bar{u}$$

as the indifference curve of a household with private wealth ω_i . That is to say, the amount of fiscal potential the household needs in order to reach utility \bar{u} in a jurisdiction with a tax rate t that receives an amount S in spillovers.

The assumptions imposed on the utility function ensure the existence and the derivability of $F^{\bar{u}}$. The slope of the indifference curve in the plane (t, F) is given by

$$F_t^{\bar{u}}(t, S, \omega_i) = \frac{1}{t} \left[\frac{\omega_i}{MRS^{\bar{u}}(t\bar{F} + S, (1-t)\omega_i)} - \bar{F} \right] \quad (7)$$

The next lemma, which will be used to prove the sufficiency of the GSC condition to have every stable jurisdiction structure segregated, states that the GSC condition implies the ordering of the indifference curves slopes with respect to the private wealth.

Lemma 4. For any preferences belonging to \mathbb{U} , $\forall(t, S) \in [0; 1] \times \mathbb{R}_+$, one has, $\forall \omega_i, \omega_k \in \mathbb{R}_+, \omega_i < \omega_k$, $\frac{\partial F^{\bar{u}}(t, S, \omega_i)}{\partial t} \leq$ (respectively \geq) $\frac{\partial F^{\bar{u}}(t, S, \omega_k)}{\partial t}$ if the public services are a gross substitute for (respectively a gross complement to) the composite private good⁴.

Proof. This lemma states that, if the GSC condition holds, then for any given amount of spillovers and any tax rate the slope of the indifference curves in the (t, F) space is monotonic with respect to the private wealth. We prove this lemma by using the definition of $F^{\bar{u}}(t, S, \omega_i)$ introduced above. The proof is provided for the gross complementary case, the gross substitutability case being symmetric. Let us assume that the public services is a gross complement to the composite private good. Then, by definition, $\forall p_Z, p_x, R \in \mathbb{R}_+$, $\frac{\partial Z^M(p_Z, p_x, R)}{\partial p_x} < 0$. Let $(t, F, S) \in [0; 1] \times \mathbb{R}_+^2$ be a certain combination of tax rate, fiscal potential and amount of spillovers, and let $(a, b) \in \mathbb{R}_+^2$ be two amounts of private wealth ($a < b$). Let us define $F(a)$ and $\omega(a)$ such that

$$Z^M\left(\frac{1}{F(a)}, \frac{1}{\omega(a)}, 1 + \frac{S}{F}\right) = tF + S$$

and

$$x^M\left(\frac{1}{F(a)}, \frac{1}{\omega(a)}, 1 + \frac{S}{F}\right) = (1-t)a$$

⁴Actually, the ordering of the indifference curve slopes with respect to the private wealth is equivalent to the GSC condition, but the implication is sufficient to prove our theorem. However, it shows that the GSC condition implies the condition identified by Westhoff to ensure the existence of an equilibrium, when households have identical preferences.

Hence, the Marginal Rate of Substitution of the public services to the composite private good, which is a function $MRS^u(Z, x)$ is equal, at the optimum, to the prices ratio:

$$MRS^u(tF + S, (1-t)a) = \frac{\omega(a)}{F(a)} \quad (8)$$

$$(9)$$

and, by definition, the chosen bundle respects the budget constraint:

$$\frac{tF + S}{F(a)} + \frac{(1-t)a}{\omega(a)} = 1 + \frac{S}{F} \quad (10)$$

Combining (8) and (10) yields:

$$\frac{1-t}{F(a) + tF + S} = \frac{MRS^u(tF + S, (1-t)a)}{a}$$

Let us now define $\omega(b)$ such that $\frac{1}{\omega(b)}$ is the highest price of the private good that would allow a household with private wealth $1 + \frac{S}{F}$ to afford the bundle (not necessarily the optimal one) $(tF + S, x)$, with $\frac{x}{\omega(b)} = (1-t)b$, if the public services price is still $\frac{1}{F(a)}$. Given the budget constraint, one has:

$$\omega(b) = \frac{F(a)(1-t)b}{F(a) - (tF + S)} > \omega(a) \quad (11)$$

Since public services are a complement to the private good, then one must have:

$$Z^M\left(\frac{1}{F(a)}, \frac{1}{\omega(a)}, 1 + \frac{S}{F}\right) \leq Z^M\left(\frac{1}{F(a)}, \frac{1}{\omega(b)}, 1 + \frac{S}{F}\right)$$

Moreover, the slope of the indifference curve must be, in absolute value, more than the price ratio $\frac{\omega(k)}{F(a)}$:

$$MRS^{\bar{u}}(tF + S, (1-t)b) \geq \frac{\omega(k)}{F(a)}$$

which is equivalent to

$$\frac{MRS^{\bar{u}}(tF + S, (1-t)b)}{b} \geq \frac{(1-t)}{F(a) - (tF + S)}$$

Using (11), one obtains:

$$\begin{aligned} \frac{MRS^{\bar{u}}(tF + S, (1-t)b)}{b} &\geq \frac{MRS^{\bar{u}}(tF + S, (1-t)a)}{a} \\ &\Leftrightarrow \\ \frac{b}{MRS^{\bar{u}}(tF + S, (1-t)b)} &\leq \frac{a}{MRS^{\bar{u}}(tF + S, (1-t)a)} \end{aligned}$$

Using the definition of $F_t^{\bar{u}}$ given by (7), the implication is established. \square

This lemma is particularly important when proving the sufficiency of this article's main result, which is the following theorem. It implies that $\forall \bar{S} \in \mathbb{R}_+$, and for any pair of jurisdictions j and j' , both receiving an amount \bar{S} of spillovers, $\forall h, k \in N$, if households h and k strictly prefer to live in jurisdiction j rather j' , then any $i \in [h, k]$ will also prefer j to j' .

Theorem 1. *For any possible economy $(\omega, U, \mathbb{L}, B, \alpha) \in \Delta$, every stable jurisdiction structure will be segregated if, and only if, U is such that the GSC condition is satisfied.*

Let us begin the proof of this theorem by the sufficiency of the condition.

Proposition 1. *For all economies $(\omega, U, \mathbb{L}, B, \alpha) \in \Delta$, if the GSC condition holds, then every stable jurisdiction structure is segregated.*

Proof. To prove this proposition, we use the lemma 4 to demonstrate that if a non-segregated jurisdictions structure arise at the equilibrium, then the GSC condition is not respected. This proof needs no assumption on how the spillovers coefficients are determined nor on the congestion parameters. Suppose that there exist 2 jurisdictions j_1 and j_2 with respective parameters (F_1, S_1, t_1) and (F_2, S_2, t_2) and 3 households $h, i, k \in N$ with private wealth $\omega_h < \omega_i < \omega_k$, such that :

$$U(t_1 F_1 + S_1, (1 - t_1)\omega_h) > U(t_2 F_2 + S_2, (1 - t_2)\omega_h)$$

$$U(t_1 F_1 + S_1, (1 - t_1)\omega_i) < U(t_2 F_2 + S_2, (1 - t_2)\omega_i)$$

$$U(t_1 F_1 + S_1, (1 - t_1)\omega_k) > U(t_2 F_2 + S_2, (1 - t_2)\omega_k)$$

Suppose, with no loss of generality, that $S_1 > S_2$. Consider the hypothetical jurisdiction j_0 with parameters (F_0, S_2, t_1) with $F_0 = \frac{t_1 F_1 + S_1 - S_2}{t_1}$.

Hence, every household is indifferent between j_1 and j_0 , because both jurisdictions offer the quantity of available public services and the same tax rate. Then,

$$U(t_1 F_0 + S_2, (1 - t_1)\omega_h) > U(t_2 F_2 + S_2, (1 - t_2)\omega_h)$$

$$U(t_1 F_0 + S_2, (1 - t_1)\omega_i) < U(t_2 F_2 + S_2, (1 - t_2)\omega_i)$$

$$U(t_1 F_0 + S_2, (1 - t_1)\omega_k) > U(t_2 F_2 + S_2, (1 - t_2)\omega_k)$$

which, according to the lemma 4, is impossible if the GSC condition holds. \square

Now that the sufficiency of the GCS condition to have all stable jurisdictions structure segregated has been proved, we look at the following proposition. It states that it is also necessary, thus any violation of the GCS condition allows one to construct a non-segregated, but yet stable, jurisdiction structure.

Proposition 2. *For all economies belonging to Δ , every stable jurisdiction structure will be segregated **only if** the GSC condition holds.*

Proof. The proof that the GSC condition is necessary in order to have all stable jurisdiction structures segregated when congestion effects and spillovers are taken into account, is close to the proof developed by Gravel and Thoron. Let us suppose that the GSC condition is violated, for some a non-degenerated interval of private wealth $[a; c]$, for some fiscal potential \bar{F} and for some amount of received spillovers \bar{S} . For this proof, we suppose that there exist two types of identical jurisdictions, with M_1 jurisdiction of type 1 and only one jurisdiction of type 2, M_1 being an integer. All jurisdictions have a fiscal potential equal to \bar{F} and receive spillovers equal to \bar{S} . The tax rate of a type l jurisdiction is denoted t_l and, finally, spillovers coefficient from a type l jurisdiction to a type l' jurisdiction is $\beta_{ll'}$.

We can prove that there always exist:

- three levels of private wealth $\omega_h, \omega_i, \omega_k$, with $\omega_h < \omega_i < \omega_k$,
- integers n_h, n_i, n_k , being the number of households of type h, i, k
- a congestion parameter $\alpha \in \mathbb{R}_+$,

- spillovers coefficients $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} \in [0; 1]$,
- and integers M_1, M_2

such that all households endowed with private wealth h and k prefer living in a jurisdiction of type 1 rather than in a jurisdiction of type 2, while all households endowed with ω_i are strictly better off in a jurisdiction of type 2 than in a jurisdiction of type 1.

As the GSC condition is violated, the favorite tax rate function is non-monotonic. Suppose, with no loss of generality, that the favorite tax rate function, for a fiscal potential equal to \bar{F} and for some amount of received spillovers \bar{S} , is decreasing with respect to the private wealth on the interval $[a, b]$ and increasing on the interval $[b; c]$. Suppose also, with no loss of generality, that $t^*(F, a) < t^*(F, c)$.

Using the property that the set of rational numbers is dense in the set of real numbers, we know that, $\forall \epsilon \in]0; b - a[$, there exists $\omega_h \in \mathbb{Q} \cap [a; b - \epsilon]$ and $\omega_i \in \mathbb{Q} \cap]b - \epsilon; b[$. As the preferred tax rate function is assumed to be decreasing on $[a; b]$, obviously one has $t^*(\bar{F}, \bar{S}, \omega_h) > t^*(\bar{F}, \bar{S}, \omega_i)$.

As the utility function is single-peaked with respect to the tax rate (lemma 1), if a household prefers a tax rate t to t' , then it also prefers any tax rate included between t and t' . Using this property, the fact that $t^*(\bar{F}, \bar{S}, \omega_h) > t^*(\bar{F}, \bar{S}, \omega_i)$, and the assumption that $t^*(\bar{F}, \bar{S}, a) < t^*(\bar{F}, \bar{S}, c)$, we know that there exists $\omega_k \in \mathbb{Q} \cap]b; c[$ such that $U(t^*(\bar{F}, \bar{S}, \omega_h)\bar{F}), (1 - t^*(\bar{F}, \bar{S}, \omega_h))\omega_k) > U(t^*(\bar{F}, \bar{S}, \omega_i)\bar{F}), (1 - t^*(\bar{F}, \bar{S}, \omega_i))\omega_k)$. Hence, a jurisdiction structure only composed of jurisdictions of type 1 and of type 2, both having a fiscal potential equal to \bar{F} , with $t_1 = t^*(\bar{F}, \bar{S}, \omega_h)$ ⁵ and $t_2 = t^*(\bar{F}, \bar{S}, \omega_i)$ ⁶, with all households endowed with private wealth ω_h or ω_k lives in j while households with private wealth ω_i live in j' , is stable. Furthermore, it is clearly non-segregated. Notice that the amount of public services produced is $\zeta_1 = t_1\bar{F}$ for a jurisdiction of type 1, and $\zeta_2 = t_2\bar{F}$ for a jurisdiction of type 2. As, per definition, $t_1 > t_2$, one has $\zeta_1 > \zeta_2$.

The major part of the proof consists of demonstrating there exist integers n_h, n_i, n_k , a congestion parameter $\alpha \in \mathbb{R}_+$, spillovers coefficients $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} \in [0; 1]$, and an integer M_1 such that every jurisdiction receives an amount of spillovers equal to \bar{S} and has a fiscal potential equal to \bar{F} .

We fix n_h, n_i, n_k such that every jurisdiction has the same population and that the average wealth are identical in both jurisdictions, thus:

$$n_h + n_k = n_i \quad (12)$$

$$n_h\omega_h + n_k\omega_k = n_i\omega_i \quad (13)$$

As $(\omega_h, \omega_i, \omega_k) \in \mathbb{Q}_+^3$, we know that there exist integers $h_1, h_2, i_1, i_2, k_1, k_2$ such that $\omega_h = \frac{h_1}{h_2}$, $\omega_i = \frac{i_1}{i_2}$ and $\omega_k = \frac{k_1}{k_2}$. Consequently, $\forall \lambda \in \mathbb{N}, n_h = \lambda h_2(k_1 i_2 - k_2 i_1)$, $n_i = \lambda i_2(h_1 k_2 + h_2 k_1)$ and $n_k = \lambda k_2(i_1 h_2 - i_2 h_1)$ are solutions satisfying (12) and (13), the fact that n_h, n_i, n_k are integers.

Now, let us prove that there always exist $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{R}_+$, $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} \in [0; 1]$, $M_1, M_2 \in \mathbb{N}$ such that:

$$\bar{F} = \frac{n_h\omega_h + n_k\omega_k}{[(n_h + n_k)(1 + (M_1 - 1)\beta_{11}) + n_i\beta_{21}]^\alpha} = \frac{n_i\omega_i}{[(n_h + n_k)M_1\beta_{12} + n_i(1 + (M_2 - 1)\beta_{22})]^\alpha} \quad (14)$$

$$\beta_{11}(M_1 - 1)\zeta_1 + M_2\beta_{12}\zeta_2 = \bar{S} \quad (15)$$

$$\beta_{21}M_1\zeta_1 + \beta_{22}(M_2 - 1)\zeta_2 = \bar{S} \quad (16)$$

⁵This would be the case if the tax rate is determined by a Lexi-min rule. A more complex proof could be provided for the majority voting rule case.

⁶Any voting rule that respects Pareto principle would lead to this tax rate, as it is the favorite tax rate of any household living in a jurisdiction of type 2.

With $\beta_{11} = \frac{\bar{S}}{(M_1-1)\zeta_1}$, $\beta_{12} = 0$, and $M_1 = \lfloor \frac{\bar{S}}{\zeta_1} \rfloor + 2$, equation 15 is satisfied, $\beta_{11} > 0$ and M_1 is an integer.

With $\beta_{21} = \frac{\bar{S}(1-\frac{\zeta_2}{\zeta_1})}{M_1\zeta_1+M_2\zeta_2}$, which is greater than 0 as $\zeta_2 < \zeta_1$, and with $\beta_{22} = \frac{\bar{S}(M_2+M_1)}{(M_2-1)(M_1\zeta_1+M_2\zeta_2)}$, which is obviously positive, equation 16 is satisfied. As $\lim_{M_2 \rightarrow +\infty} \beta_{21} = 0$ and $\lim_{M_2 \rightarrow +\infty} \beta_{22} = 0$, one can ensure that $\beta_{21} < 1$ and $\beta_{22} < 1$ by assigning a sufficiently high integer value to M_2 .

Furthermore, with the values selected for $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$, the left part of equation (14) is trivial, as the numerators are equal, and so are the denominators.

The final step of the proof is to find $\lambda \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+$ such that

$$\frac{n_i \omega_i}{[(n_h + n_k)M_1\beta_{12} + n_i(1 + (M_2 - 1)\beta_{22})]^\alpha} = \bar{F}$$

Replacing n_h, n_i and n_k by their attributed value, one obtains:

$$\frac{\lambda i_2(h_1k_2 + h_2k_1)\omega_i}{[\lambda i_2(h_2k_1 - h_1k_2)M_1\beta_{12} + \lambda i_2(h_1k_2 + h_2k_1)(1 + (M_2 - 1)\beta_{22})]^\alpha} = \bar{F}$$

Using the natural logarithm function, and after some algebra, one has:

$$\alpha = 1 + \frac{\log(i_2(h_1k_2 + h_2k_1)\omega_i) - \log(\bar{F})}{\log(\lambda) + \log[i_2(h_2k_1 - h_1k_2)M_1\beta_{12} + i_2(h_1k_2 + h_2k_1)(1 + (M_2 - 1)\beta_{22})]}$$

As $\lim_{\lambda \rightarrow +\infty} 1 + \frac{\log(i_2(h_1k_2 + h_2k_1)\omega_i) - \log(\bar{F})}{\log(\lambda) + \log[i_2(h_2k_1 - h_1k_2)M_1\beta_{12} + i_2(h_1k_2 + h_2k_1)(1 + (M_2 - 1)\beta_{22})]} = 1$, one can ensure that, even in the case where $\log(i_2(h_1k_2 + h_2k_1)\omega_i) - \log(\bar{F}) < 0$, there exists $\alpha > 0$, by selecting a sufficiently high integer value for λ .

Then, for any violation of the monotonicity of the preferred tax rate function with respect to the private wealth, one can always construct a stable and yet non-segregated jurisdiction structure. \square

5 Conclusion

The conclusion of this paper is that neither the existence of spillovers across jurisdictions, nor the congestion modify the necessity or the sufficiency of the GSC condition to ensure the segregation of every stable jurisdiction structure in a model *a la* Westhoff.

However, this result does not imply that introducing congestion or spillovers into a model *a la* Westhoff would have no impact on the stability or on the segregative properties of endogenous jurisdiction structure formation, as proven in the example provided in section 3.

The presence of congestion effects obviously mitigates the existence of an equilibrium. Consequently, stronger conditions must be found in order to ensure that a stable jurisdiction structure will arise.

This condition has been proven robust regarding several generalizations of the model. Searching for a generalization that would make the condition either too weak or too strong to have all stable jurisdictions structures would be an interesting topic for further researches.

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6 Appendix

6.1 Proof Lemma 1

The proof of this lemma can be easily obtained by showing that the utility function is concave with respect to t , so there exists a unique

$$t^* \in \arg \max_{t \in [0;1]} U(tF + S, (1-t)\omega_i)$$

Let us denote

$$g_{ij} : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ t & \longmapsto g_{ij}(t) \end{cases} = U(tF_j + S_j; (1-t)\omega_i)$$

as the utility of a household i , living in jurisdiction j endowed with of a fiscal potential F_j and receiving an amount S_j of spillovers, depending on the tax rate.

The first derivative of this function with respect to t is:

$$g'_{ij}(t) = F_j U_Z - \omega_i U_x$$

So, one has:

$$g''_{ij}(t) = F_j^2 U_{ZZ} - 2F_j \omega_i U_{Zx} + \omega_i^2 U_{xx}$$

This expression can be obtained by pre-multiplying and post-multiplying the utility function's Hessian matrix by the vector $[F_j, \omega_i]$. Since $U(Z, x)$ is concave, its Hessian matrix is a negative semi-definite matrix. Hence, by the definition of a negative semi-definite matrix, the whole expression is non-positive⁷, which means that the utility function is weakly concave with respect to the tax rate. As a consequence, the utility function is single-peaked with respect to the tax rate on the interval $[0; 1]$, so there exists a unique $t^* \in [0; 1]$.

6.2 With no spillovers and no congestion effects

With no spillovers and no congestion effects, households of type a and c living in j_1 , and households of type b living in j_2 would lead to an aggregate wealth close to 525,650 in j_1 , and equal to 450,000 in j_2 . The tax rate in j_1 , denoted t_1 , would be equal to $\frac{1}{2}$, as households of type a hold the majority, while $t_2 = \frac{1}{3}$. One would have $Z_1 \approx 262,825$ and $Z_2 = 150,000$. Such a jurisdiction structure is stable, since every household has a greater utility in its jurisdiction than it would if it were to move to the other jurisdiction, as indicated by the following figure:

	j_1	j_2
a	13.565	13.328
b	14.917	14.918
c	15.479	15.474

6.3 With congestion effects but no spillovers

Once one introduces congestion effects, the amount of public services in j_1 will be approximately $Z_1 \approx 12,947$ and, in j_2 , $Z_2 \approx 12,041$ which will lead households of type c to move to j_2 , as the new utilities are:

	j_1	j_2
a	10.824	10.806
b	12.175	12.396
c	12.738	12.952

Will the new jurisdiction structure be stable after households of type c have moved from j_1 to j_2 ? Since the preferred tax rate function is constant with respect to the fiscal potential, the tax rate will be the same in every jurisdiction.

Once households of type c have moved from j_1 to j_2 , the amount of public services in j_1 will be $Z_1 \approx 16,941$ and in j_2 , $Z_2 \approx 12,046$. Therefore, households of

⁷See for instance [10]

type a will get a higher utility level in j_1 than in j_2 , while households of type b and of type c can enjoy a higher utility level by staying in j_2 than if they were to move to j_1 :

	j_1	j_2
a	10.823	10.806
b	12.175	12.396
c	12.737	12.952

This new structure is stable and segregated, while the previous one was stable and non-segregated as long as no congestion effects were assumed.

6.4 With congestion effects and spillovers

With such spillover coefficients, the available amounts of public services households can enjoy are now $Z_1 \approx 24,970$ and $Z_2 \approx 14,263$. As a consequence, households of type c will move back to j_1 :

	j_1	j_2
a	11.211	10.975
b	12.563	12.565
c	13.125	13.121

Finally, the jurisdiction structure in which jurisdiction j_1 is composed of households of type a and c , and j_2 is composed of households of type b is stable: with households of type c in j_1 instead of j_2 , $Z_1 \approx 24,972$ and $Z_2 \approx 14,261$.

One can observe that, due to the existence of spillovers between jurisdictions, households of type c 's change of jurisdiction have almost no impact on the available amount of public services in each jurisdiction. Therefore, the utility levels will remain almost the same for all types of households.